CHAPTER

8

Quadratic graphs

What you will learn

Rockets

The National Aeronautics and Space Administration (NASA) in the USA has sent many rockets into space, as have other countries such as Russia and France. Some of these rockets deploy satellites which travel our solar system collecting information about other planets. The path that these travel when leaving earth forms a parabolic shape, and the equation used to model it involves x^2. This is due to the way gravity acts on moving objects. The distance that things fall depends on the square of the time for which they are falling.

VELS

Number

- Use calculators for arithmetic computations and rational approximations of irrational numbers
- Carry out exact arithmetic computations involving square roots of prime numbers and rational numbers that are not perfect squares
- Carry out computations to a required accuracy in terms of decimal place

Structure

- Apply algebraic properties to rearrange formulas, rearrange and simplify algebraic expressions, and verify the equivalence of algebraic expressions
- Identify quadratic functions by table, rule and graph
- Recognise and explain the roles of the constants in $f(x) = a(x + b)^2 + c$
- Solve equations of the form $f(x) = k$, where k is a real constant, and simultaneous linear and quadratic equations

Working mathematically

- Formulate generalisations and arguments in natural language and symbolic form for quadratic functions.
- Use technology to analyse the quadratic function
- Represent relationships in mathematical terms of the features of the context being modelled
- Routinely make judgements about the reasonableness of calculations

Do now

1 Factorise the following.

a $x^2 - 16$ **b** $121x^2 - 36$
c $x^2 + 5x + 4$ **d** $x^2 - 9x + 20$
e $x^2 + 15x - 16$ **f** $3x^2 - 21x + 36$

2 Factorise the following by completing the square.

a $x^2 + 8x + 1$ **b** $3x^2 - 12x - 6$

3 Solve the following quadratic equations.

a $x^2 - 3x = 0$ **b** $x^2 - 8x + 16 = 0$

4 Use the discriminant ($b^2 - 4ac$) to determine the number of solutions to the following equations.

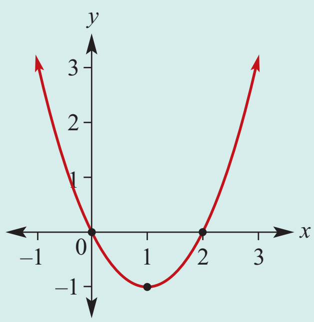
a $3x^2 - 2x + 4 = 0$ **b** $x^2 + 3x - 2 = 0$ **c** $3x^2 + 6x + 3 = 0$

5 Solve the following using the quadratic formula.

a $x^2 + 9x + 1 = 0$ **b** $3x^2 - 7x + 2 = 0$

6 For the graph state the coordinates of the:

a x intercepts
b y intercept
c turning point



7 For the following functions:

a $f(x) = 2x + 3$ **b** $f(x) = -3x - 5$ **c** $f(x) = 3$

Determine:

i $f(0)$ **ii** $f(-1)$ **iii** $f(5)$ **iv** $\{x: f(x) = 0\}$

Answers

1 a $(x-4)(x+4)$ **b** $(11x-6)(11x+6)$ **c** $(x+4)(x+1)$ **d** $(x-5)(x-4)$ **e** $(x+16)(x-1)$ **f** $3(x-3)(x-4)$ **2 a** $(x+4-\sqrt{15})(x+4+\sqrt{15})$ **b** $3(x-2-\sqrt{6})(x-2+\sqrt{6})$ **3 a** $x = 3$ or $x = 0$ **b** $x = 4$ **4 a** 0 solutions **b** 2 solutions **c** 1 solution **5 a** $x = \dfrac{-9 \pm \sqrt{77}}{2}$ **b** $x = 2$ or $-\frac{1}{3}$

6 a (0, 0) and (2, 0) **b** (0, 0) **c** (1, −1) **7 a i** $f(0) = 3$ **ii** $f(-1) = 1$ **iii** $f(5) = 13$ **iv** $-\frac{3}{2}$ **b i** $f(0) = -5$ **ii** $f(-1) = -2$ **iii** $f(5) = -20$ **iv** $-\frac{5}{3}$ **c i** $f(0) = 3$ **ii** $f(-1) = 3$ **iii** $f(5) = 3$ **iv** no solution

8.1 Plotting quadratic graphs

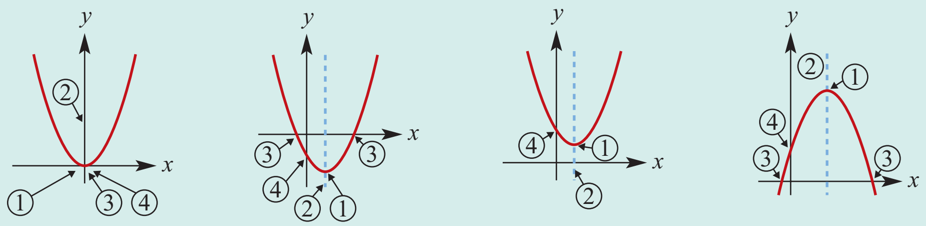
Linear relations that were dealt with previously took the form of $y = mx + c$, where m, c were constants. With these graphs we were able to see some key features, including gradient, x intercepts and y intercepts. Drawing the set of points equidistant from a point and a line produces a new type of graph. It is a smooth continuous curve called a parabola, based on the quadratic relation $y = x^2$.

A quadratic relation is a function, hence we can use the notation $y = \ldots$ or $f(x) = \ldots$

Parabolas can be sketched over a given set of x values (the domain) which produces a set of y values (the range).

Key ideas

- **Quadratic relations**, now defined as $f(x) = ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$, produce graphs called **parabolas**.



- The key features of a parabola are:
  1 the turning point: a maximum or a minimum
  2 the axis of symmetry
  3 the x intercepts (can be 0, 1 or 2 intercepts)
  4 the y intercept
- For endpoints, use an open circle for non-inclusive and a closed circle for inclusive.
- Set notation, e.g. $\{x: -1 < x \leqslant 2\}$, or interval notation, e.g. $(-1, 2]$, can be used to describe domain and range.

Example 1

Determine the following key features of each of the given graphs.

i turning point and its nature **ii** axis of symmetry

iii x-intercepts **iv** y-intercept

Solution

a **i** turning point is a minimum at $(1, -4)$
From the graph we can see the lowest point at $(1, -4)$

ii axis of symmetry is $x = 1$
Describe the central vertical line

iii x intercepts at $(-1, 0)$ and $(3, 0)$

iv y intercept at $(0, -3)$
The intercepts are read from the x and y axes

b **i** Turning point is a maximum at $(-2, 0)$
From the graph we see the highest point at $(-2, 0)$

ii axis of symmetry is $x = -2$
Describe the central vertical line

iii x intercept at $(-2, 0)$

iv y intercept at $(0, -4)$
There is only one x intercept, as the turning point is on the x axis

Example 2

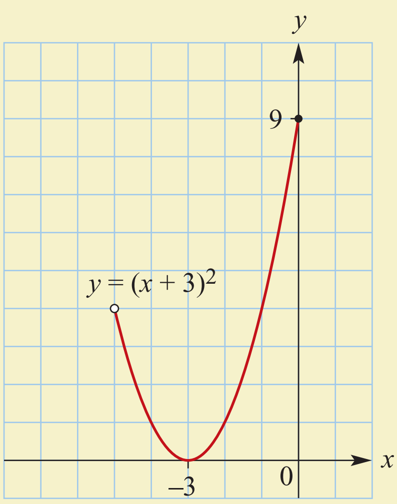
Using graph paper, plot the graph of $f(x) = (x + 3)^2$ for $-5 < x \leqslant 0$ and determine the:

a turning point and its nature
b axis of symmetry
c x intercept(s)
d y intercept
e range

Solution

x	-5	-4	-3	-2	-1	0
$f(x)$	4	1	0	1	4	9

Draw up a table of values



Plot the points and join them with a smooth curve

Use an open circle to show that the point $(-5, 4)$ is not included, and a closed circle to show that the point $(0, 9)$ is included

a turning point is a minimum at $(-3, 0)$
Read off the turning point

b axis of symmetry $x = -3$
Determine the equation of the axis of symmetry from the turning point

c x intercept $(-3, 0)$
Read off the x intercept

d y intercept $(0, 9)$
Read off the y intercept

e range $= [0, 9]$
The resulting y values range from 0 to 9 inclusive

Example 3

Plot the graph of $y = -x^2 + 4x - 1$ over the domain $-2 \leqslant x \leqslant 6$ using technology and determine:

a the turning point and its nature

b the axis of symmetry

c the x intercept(s)

d the y intercept

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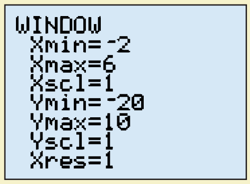
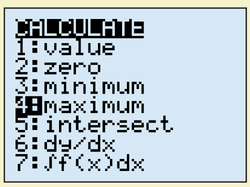
Go to **WINDOW** and set **Xmin** at -2 and **Xmax** at 6, **Ymin** at -20 and **Ymax** at 10.



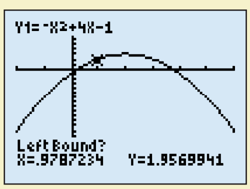
Go to **Y =** and type $-x^2 + 4x - 1$.

To plot press **GRAPH**.

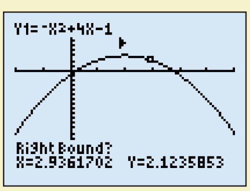
a Decide whether the turning point is a maximum or a minimum. Press **2nd CALC maximum**.



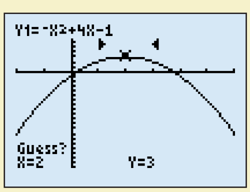
Arrow to the left of the turning point (called the left bound), press ENTER.



Arrow to the right of the turning point (called the right bound), press ENTER.



Arrow to a point close to the turning point and press ENTER.



Turning point is a maximum at (2, 3).

b Axis of symmetry is $x = 2$.

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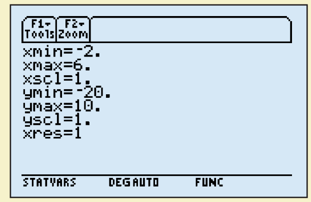
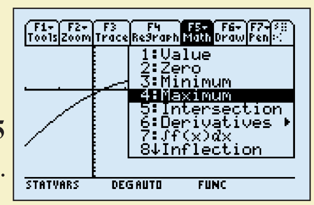
Go to **WINDOW** and set **xmin** at -2 and **xmax** at 6, **ymin** at -20 and **ymax** at 10



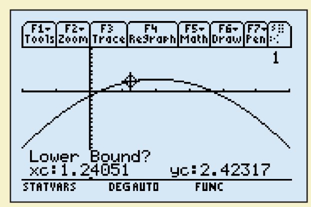
Go to ◆**Y =** and type $-x^2 + 4x - 1$.

To plot press **GRAPH**.

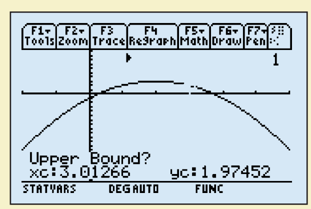
a Decide whether the turning point is a maximum or a minimum. Press **F5 MATH Maximum**.



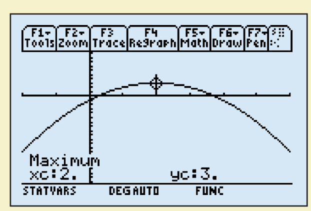
Arrow just to the left of the turning point (called the lower bound), press ENTER.



Arrow just to the right of the turning point (called the upper bound), press ENTER.

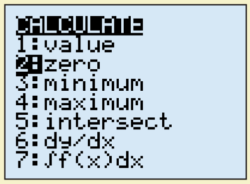


Turning point is a maximum at (2, 3).



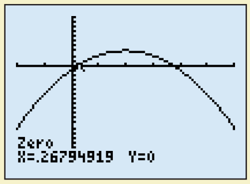
b Axis of symmetry is $x = 2$.

c For the x-intercepts press **2nd CALC zero**. Arrow to the left of the intercept, press ENTER.



Arrow to the right of the intercept, press ENTER.

Arrow to a point close to the x intercept and press ENTER.

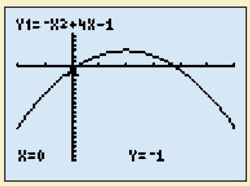


Repeat the process for the other intercept.

x intercepts at (0.268,0) and (3.732.0)

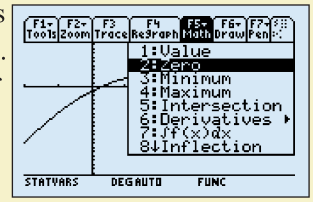
d For the y intercept press **2nd CALC value**.

Press **0** then ENTER.



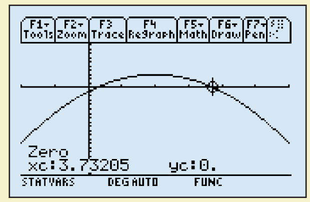
y intercept at (0, -1).

c For the x-intercepts press **F5 Math Zero**. Arrow to the left of the intercept, press ENTER.



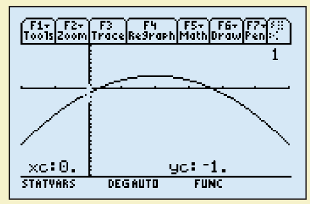
Arrow to the right of the intercept.

Press ENTER. Repeat the process for the other intercept.



x intercepts at (0.268,0) and (3.732.0)

d For the y intercept press **F5 Math Value**. Press **0** then ENTER.



y intercept at (0, -1).

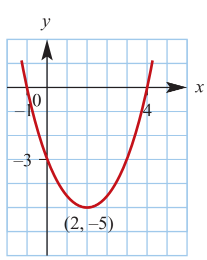
Exercise 8A

Example 1

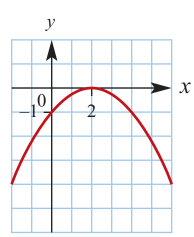
1 Determine the key features of the following graphs.

i turning point **ii** axis of symmetry
iii x intercepts **iv** y intercept

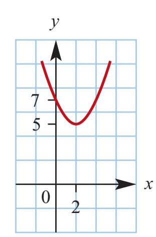
a



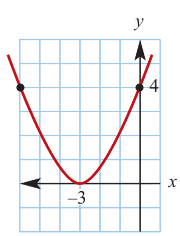
b



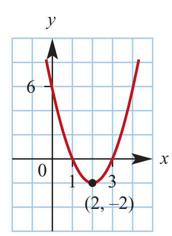
c



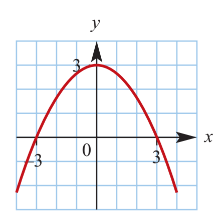
d



e



f



Example 2

2 Using graph paper, plot the graph of the following over the given domain and determine the:

i turning point and its nature **ii** axis of symmetry
iii x intercepts **iv** y intercept
v range

a $f(x) = x^2$ $\quad -4 \leqslant x \leqslant -4$
b $f(x) = x^2 + 3$ $\quad -5 \leqslant x \leqslant 2$
c $f(x) = (x - 3)^2$ $\quad -2 \leqslant x \leqslant 5$
d $f(x) = (x - 2)^2 + 1$ $\quad -2 \leqslant x \leqslant 4$
e $f(x) = (x + 3)^2 - 4$ $\quad -5 \leqslant x \leqslant 1$
f $f(x) = 2(x - 2)^2 + 1$ $\quad -2 \leqslant x \leqslant 5$
g $f(x) = 3(x + 2)^2 - 3$ $\quad -4 \leqslant x \leqslant 2$
h $f(x) = -3x^2 + 12$ $\quad -3 \leqslant x \leqslant 3$
i $f(x) = -4(x - 2)^2$ $\quad -1 \leqslant x \leqslant 5$
j $f(x) = -2(x + 3)^2 + 2$ $\quad -5 \leqslant x \leqslant 0$

Example 3

3 Using appropriate technology (for example, a graphics or CAS calculator) or otherwise, plot the following graphs over the specified domain and determine:

i turning point **ii** axis of symmetry
iii x intercepts **iv** y intercept

a $y = x^2 + 5x + 4$ $\quad -5 \leqslant x \leqslant 1$
b $y = x^2 + 2x - 3$ $\quad -4 \leqslant x \leqslant 4$
c $y = x^2 - 3x$ $\quad -1 \leqslant x \leqslant 4$
d $y = x^2 + 5x$ $\quad -10 \leqslant x \leqslant 5$
e $y = -x^2 - x + 6$ $\quad -4 \leqslant x \leqslant 3$
f $y = -x^2 - 5x + 14$ $\quad -7 \leqslant x \leqslant 1$
g $y = -x^2 + 4x$ $\quad -1 \leqslant x \leqslant 6$
h $y = -x^2 - 5x$ $\quad -8 \leqslant x \leqslant 2$

4 For $f(x) = x^2 + 3x + 7$ find the following. Note: $f(a)$ means 'find the function value (y value) at $x = a$'.

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(2)$
e $f(4)$ **f** $f(6)$ **g** $f(-2)$ **h** $f(-4)$

5 Find the coordinates of the end points of $f(x) = 2(x - 1)^2 - 4$ for the given domains and state the range using interval notation, e.g. (1, 7].

a $2 \leqslant x \leqslant 5$ **b** $-2 \leqslant x \leqslant 0$ **c** $3 < x \leqslant 6$ **d** $-5 < x < 1$

6 Find the maximum value of $f(x)$ if:

a $f(x) = x^2$ $\quad -2 \leqslant x \leqslant -4$
b $f(x) = x^2 + 3$ $\quad 1 \leqslant x \leqslant 2$
c $f(x) = (x - 2)^2 + 1$ $\quad 0 \leqslant x \leqslant 1$
d $f(x) = -3x^2 + 12$ $\quad 2 \leqslant x \leqslant 4$
e $f(x) = -4(x - 2)^2$ $\quad -3 \leqslant x \leqslant 1$

Enrichment

Th

7 The range can be determined using the endpoints or between one of the endpoints and the graph's maximum or minimum depending on its domain.
Sketch the following parabolas, labelling their endpoints and determine their range over the given domains.

a $y = x^2 - 5x + 4$
i $3 < x \leqslant 4$ **ii** $1 \leqslant x < 4$ **iii** $0 < x < 3$

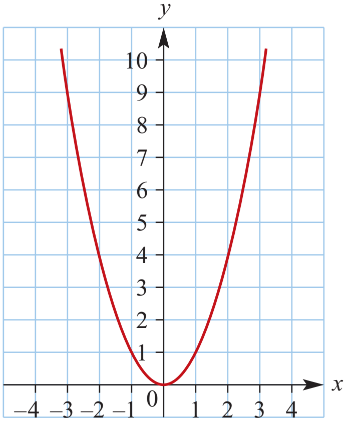
b $f(x) = x^2 - 2$
i $-5 \leqslant x \leqslant 2$ **ii** $-5 \leqslant x < -2$ **iii** $1 < x \leqslant 4$

c $f(x) = -2(x + 3)^2 + 2$
i $-5 \leqslant x < 5$ **ii** $-2 \leqslant x \leqslant 0$ **iii** $3 < x \leqslant 5$

8.2 Investigating the transformations of $f(x) = x^2$

The simplest quadratic function is $f(x) = x^2$. The graph of this function is shown.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9



It is a symmetrical continuous curve that has a minimum turning point at (0, 0).

While retaining the properties of a parabola, we can change (transform) the shape and position of the graph of $f(x) = x^2$ to form other quadratic graphs.

Key ideas

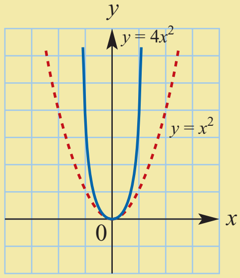
The graph of $f(x) = x^2$ can be transformed by:

- dilation—making the graph appear wider or narrower
- translation in the x and y direction—shifting left or right and up or down
- reflection—in the x and y axes

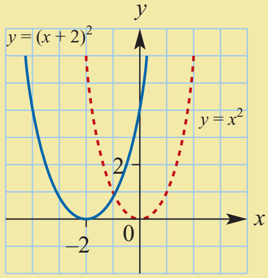
Example 4

Copy and complete the table for the following graphs.

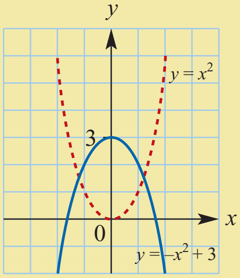
a



b



c



	Formula	Maximum or minimum	Reflected in the x axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$					
b	$y = (x + 2)^2$					
c	$y = -x^2 + 3$					

Solution

	Formula	Maximum or minimum	Reflected (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$y = 4x^2$	minimum	no	(0, 0)	4	narrower
b	$y = (x + 2)^2$	minimum	no	(−2, 0)	9	same
c	$y = -x^2 + 3$	maximum	yes	(0, 3)	2	same

Explanation

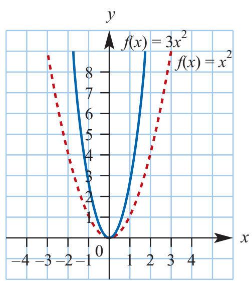
Read all information from the graph

Exercise 8B

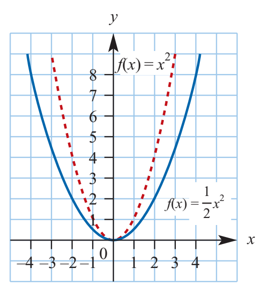
Example 4a

1 Copy and complete the table below for the following graphs.

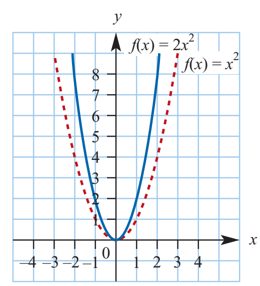
a



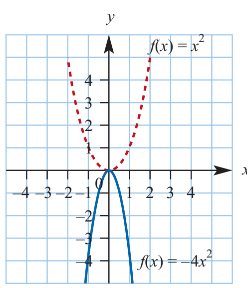
b



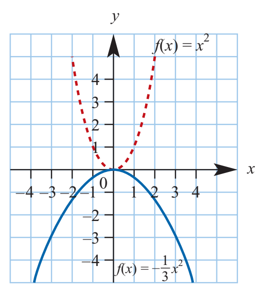
c



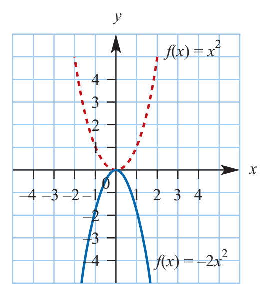
d



e



f



	Formula	Maximum or minimum	Reflected in the x axis (yes/no)	Turning point	y value when $x = 1$	Wider or narrower than $y = x^2$
a	$f(x) = 3x^2$					
b	$f(x) = \frac{1}{2}x^2$					
c	$f(x) = 2x^2$					
d	$f(x) = -4x^2$					
e	$f(x) = -\frac{1}{3}x^2$					
f	$f(x) = -2x^2$					

2 **a** Using technology, plot the following pairs of graphs on the same axis for $-5 \leqslant x \leqslant 5$ and compare their tables of values.

i $f(x) = x^2$ and $f(x) = 4x^2$

ii $f(x) = x^2$ and $f(x) = \frac{1}{3}x^2$

iii $f(x) = x^2$ and $f(x) = 6x^2$

iv $f(x) = x^2$ and $f(x) = \frac{1}{4}x^2$

v $f(x) = x^2$ and $f(x) = 7x^2$

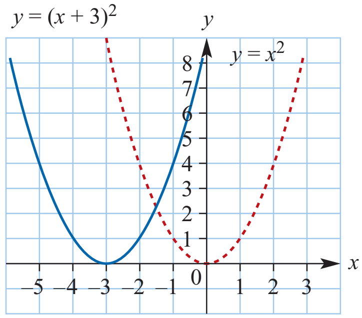
vi $f(x) = x^2$ and $f(x) = \frac{2}{5}x^2$

b Suggest how the constant a in $f(x) = ax^2$ transforms the graph of $f(x) = x^2$.

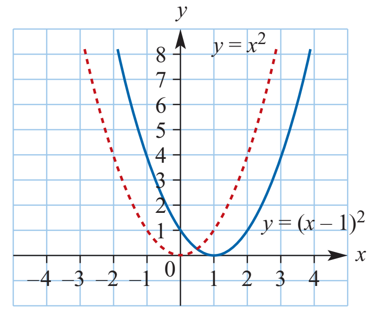
Example 4b

3 Copy and complete the table below for the following graphs.

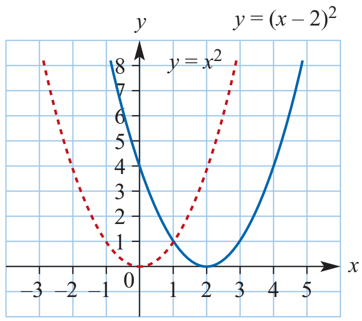
a



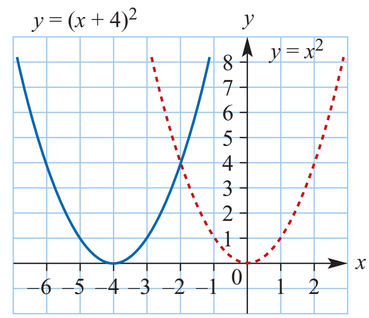
b



c



d



	Formula	Turning point	y intercept
a	$y = (x + 3)^2$		
b	$y = (x - 1)^2$		
c	$y = (x - 2)^2$		
d	$y = (x + 4)^2$		

4 **a** Using technology, plot the following sets of graphs on the same axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $f(x) = x^2$, $f(x) = (x + 1)^2$, $f(x) = (x + 2)^2$, $f(x) = (x + 3)^2$

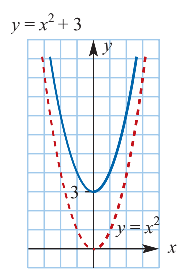
ii $f(x) = x^2$, $f(x) = (x - 1)^2$, $f(x) = (x - 2)^2$, $f(x) = (x - 3)^2$

b Explain how the constant h in $f(x) = (x + h)^2$ transforms the graph of $f(x) = x^2$.

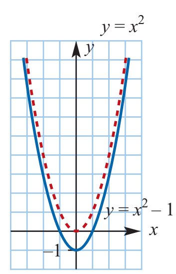
Example 4c

5 Copy and complete the table for the following graphs.

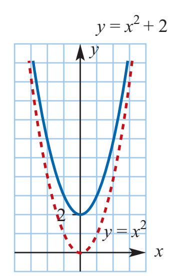
a



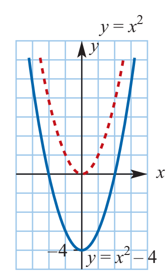
b



c



d



	Formula	Turning point	y value when $x = 1$
a	$y = x^2 + 3$		
b	$y = x^2 - 1$		
c	$y = x^2 + 2$		
d	$y = x^2 - 4$		

6 **a** Using technology, plot the following sets of graphs on the same axes for $-5 \leqslant x \leqslant 5$ and compare the turning point of each.

i $f(x) = x^2$, $f(x) = x^2 + 1$, $f(x) = x^2 + 2$, $f(x) = x^2 + 3$

ii $f(x) = x^2$, $f(x) = x^2 - 1$, $f(x) = x^2 - 3$, $f(x) = x^2 - 5$

b Explain how the constant k in $f(x) = x^2 + k$ transforms the graph of $f(x) = x^2$.

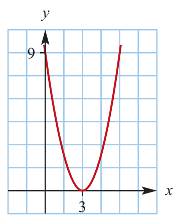
7 Match each of the following equations to one of the graphs below.

a $y = 2x^2$ **b** $y = x^2 - 6$ **c** $f(x) = (x + 2)^2$

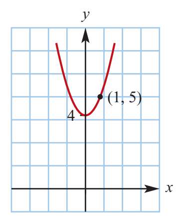
d $y = -5x^2$ **e** $f(x) = (x - 3)^2$ **f** $y = \frac{1}{4}x^2$

g $f(x) = x^2 + 4$ **h** $y = (x - 2)^2 + 3$

i



ii



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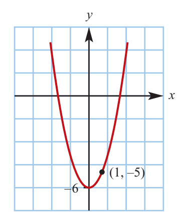

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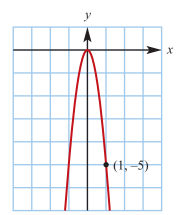

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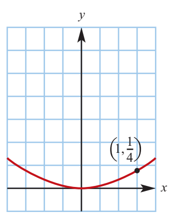

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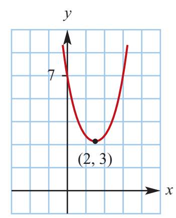

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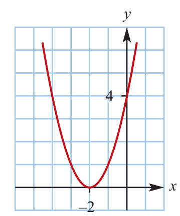

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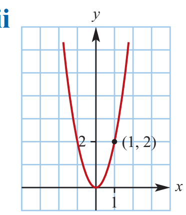


Enrichment

Th

8 Given the following functions:

i $f(x) = x^2$ **ii** $f(x) = x^2 + 4$ **iii** $f(x) = (x - 2)^2$ **iv** $f(x) = 3x^2$

write the following and describe the transformations on the original graph.

a $f(x - 1)$ **b** $f(x) + 3$ **c** $2f(x)$ **d** $-f(x)$

e $f(-x)$ **f** $f(x - h)$ **g** $f(x) + k$

9 In chapter 5, mapping notation was used to describe transformations.

- $(x, y) \longrightarrow (x + 1, -y)$ means a translation in the positive x direction and a reflection in the x axis.

Find the equation of the image of $f(x) = x^2$ after the following transformations.

a $(x, y) \longrightarrow (x + 1, y - 1)$ **b** $(x, y) \longrightarrow (-x, -y)$

c $(x, y) \longrightarrow (2x, y + 2)$ **d** $(x, y) \longrightarrow \left(\dfrac{x - 3}{2}, 3y\right)$

8.3 Sketching with transformations

From the previous exercise, you developed some useful generalities about the graphs of quadratic functions that undergo single transformations on $f(x) = x^2$.

Combined transformations include more than one of the individual transformations of the function. Functions of the form $f(x) = a(x - h)^2$ will be considered.

Key ideas

- When sketching a parabola it can be helpful to perform transformations in the following order.
  - dilations and reflections first
  - translations last
- Show the following key features on a graph:
  - its shape—to show maximum or minimum
  - the turning point—to show the translation of $y = x^2$
  - any other point (usually the y intercept or when $x = 1$ if the y intercept is at the turning point)—to show the dilation of $y = x^2$

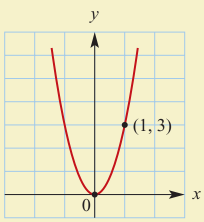
Example 5

Sketch graphs of the following quadratic functions.

a $f(x) = 3x^2$ **b** $f(x) = -x^2 + 4$ **c** $f(x) = (x - 2)^2$

Solution

a



Explanation

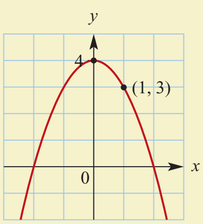
Decide on the shape

Determine the turning point: (0, 0)

Find y when $x = 1$: $y = 3(1)^2 = 3$

Sketch the graph

b



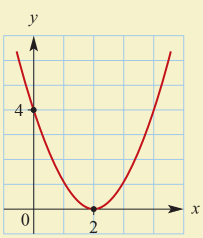
Decide on the shape

Determine the turning point: (0, 4)

Find y when $x = 1$: $y = -(1)^2 + 4 = 3$

Sketch the graph

c



Decide on the shape

Determine the turning point: (2, 0)

Determine the y intercept

$x = 0$: $y = (-2)^2 = 4$

Sketch the graph

Exercise 8C

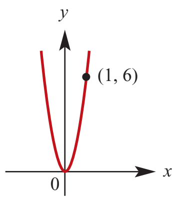
Example 5

1 Sketch graphs of the following quadratic functions.

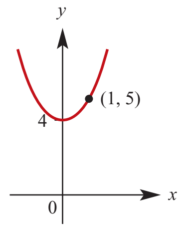
a $f(x) = 2x^2$	**b** $f(x) = -3x^2$	**c** $f(x) = \frac{1}{2}x^2$
d $f(x) = -\frac{1}{3}x^2$	**e** $f(x) = x^2 + 2$	**f** $f(x) = x^2 - 4$
g $f(x) = -x^2 + 1$	**h** $f(x) = -x^2 - 3$	**i** $f(x) = (x + 3)^2$
j $f(x) = (x - 1)^2$	**k** $f(x) = -(x + 2)^2$	**l** $f(x) = -(x - 3)^2$

2 Determine the rule for the following parabolas.

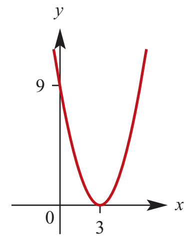
a



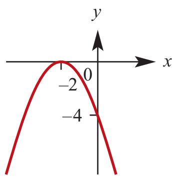
b



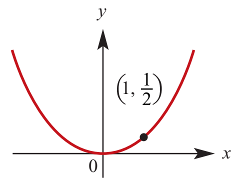
c



d



e



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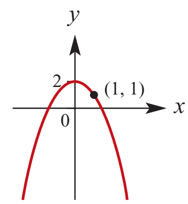
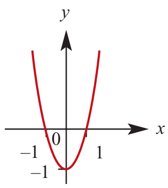
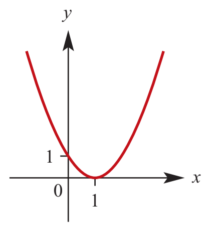


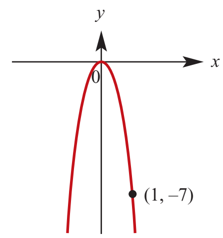
g



h

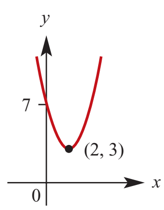


i

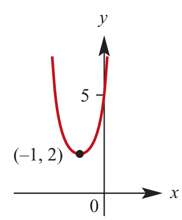


3 Consider the following graphs of the form $y = (x - h)^2 + k$ and $y = -(x - h)^2 + k$.

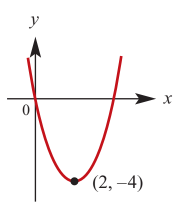
i $y = (x - 2)^2 + 3$



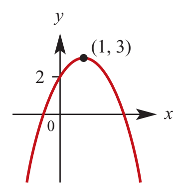
ii $y = (x + 1)^2 + 2$



iii $y = (x - 2)^2 - 4$



iv $y = -(x - 1)^2 + 3$



a State the turning point of each graph.
b How does the turning point relate to the values of h and k in the equations given initially?

4 Sketch graphs of the following functions, showing the turning points and y intercept.

a $f(x) = (x + 1)^2 + 1$ b $f(x) = (x + 2)^2 - 1$ c $f(x) = (x + 3)^2 + 2$
d $f(x) = (x - 1)^2 + 2$ e $f(x) = (x - 4)^2 + 1$ f $f(x) = (x - 1)^2 - 4$
g $f(x) = -(x + 1)^2 + 3$ h $f(x) = -(x - 2)^2 + 1$ i $f(x) = -(x + 3)^2 - 2$
j $f(x) = 4 - (x + 1)^2$ k $f(x) = 8 - (x - 2)^2$ l $f(x) = -3 - (x + 3)^2$

5 Write rules that result from the following transformations of $y = x^2$.

a Translation of 2 units left and 3 units up.
b Translation of 2 units right and 4 units up.
c Reflection in the x axis, translation of 1 unit right and 3 units up.
d Reflection in the x axis, translation of 4 units left and 2 units down.
e Translation of 6 units right.
f Translation of 4 units down.
g Reflection in the x axis.
h Reflection in the x axis, translation of 8 units down.
i Reflection in the x axis, translation of 3 units right.

6 Explain how the graph of $y = x^2$ is transformed by first determining the shape and the coordinates of the turning points of the following. The first one is done for you.

a $y = -(x - h)^2 + k$
The turning point is a maximum at (h, k).
$\therefore y = x^2$ is reflected in the x axis, translated h units right and k units up.
b $y = (x + h)^2 + k$ c $y = -(x + h)^2 + k$ d $y = (x + h)^2 - k$
e $y = (x - h)^2 + k$ f $y = -(x - h)^2 + k$ g $y = -(x - h)^2 - k$

Th

Enrichment

7 Sketch graphs of $y = x^2$ after the following transformations in the order given.

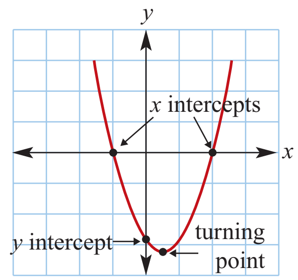
a i Translation of 2 units left then 3 units up.
ii Translation of 3 units left then 2 units up.
b i Translation of 2 units left followed by reflection in the x axis.
ii Reflection in the x axis followed by a translation of 2 units left.
c i Translation of 3 units up followed by a reflection in the y axis.
ii A reflection in the y axis followed by a translation of 3 units up.

8 Sketch the following graphs of the type $f(x) = a(x - h)^2 + k$ where $a \neq 1$

a $f(x) = 2(x - 3)^2 + 4$ b $3(x + 2)^2 + 5$
c $f(x) = -2(x - 3)^2 + 4$ d $f(x) = -2(x + 3)^2 - 4$
e $f(x) = \frac{1}{2}(x - 3)^2 + 4$ f $f(x) = -\frac{1}{2}(x - 3)^2 + 4$

8.4 Sketching quadratic graphs using factorisation

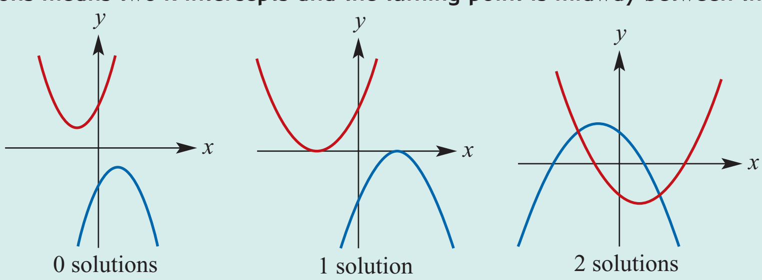
Quadratic functions of the form $f(x) = ax^2 + bx + c$ that factorise by inspection or by using a multiplication table can be graphed by considering their x intercepts. The turning point can then be found using the x intercepts.



Key ideas

Follow these steps to sketch a quadratic function of the form $f(x) = ax^2 + bx + c$ that factorises by inspection or multiplication table.

- Determine the y intercept by substituting $x = 0$.
  $f(x) = ax^2 + bx + c$ becomes $y = c$.
- Determine the x intercept by setting $f(x)$ or y equal to zero, factorising if possible, and using the null factor law.
  - 0 solutions means no x intercepts.
  - 1 solution means one x intercept and it will occur at the turning point.
  - 2 solutions means two x intercepts and the turning point is midway between them



- Determine the turning point by finding the x coordinate then substitute the x coordinate into the equation $f(x) = ax^2 + bx + c$ to determine the y coordinate.

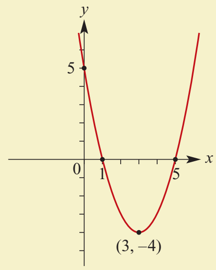
Example 6

Sketch the graph of the quadratic function $f(x) = x^2 - 6x + 5$.

Solution	Explanation
y intercept at $x = 0$: $y = 5$	y intercept at $y = c$
x intercepts at $y = 0$: $0 = x^2 - 6x + 5$	x intercept at $y = 0$
$= (x - 5)(x - 1)$	Factorise and apply the null factor law
$\therefore x = 5$ or 1	
x intercepts at $(1, 0)$ and $(5, 0)$	

Turning point at $x = 3$: $y = (3)^2 - (6 \times 3) + 5$
$= 9 - 18 + 5$
$= -4$

Turning point at $(3, -4)$



Turning point at $x = \dfrac{1 + 5}{2} = 3$

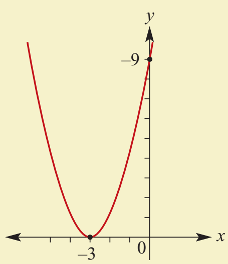
Sketch the graph showing the key points

Example 7

Sketch the graph of the quadratic function $y = x^2 + 6x + 9$.

Solution

y intercept at $x = 0$: $y = 9$
x intercepts at $y = 0$: $0 = x^2 + 6x + 9$
$= (x + 3)^2$
$\therefore x = -3$
x intercept at $(-3, 0)$
Turning point at $(-3, 0)$



Explanation

y intercept at $y = c$
x intercept at $y = 0$
Factorise and apply the null factor law

Turning point is at the x intercept since there is only one x intercept

Sketch the graph showing the key points

Exercise 8D

Example 6

1 Sketch the graphs of the following quadratic functions.

a $f(x) = x^2 - 6x + 8$ **b** $f(x) = x^2 + 6x + 8$ **c** $f(x) = x^2 + 10x + 16$
d $f(x) = x^2 - 6x - 16$ **e** $f(x) = x^2 + 2x - 8$ **f** $f(x) = x^2 - 4x - 21$
g $f(x) = x^2 + 4x + 3$ **h** $f(x) = x^2 - 8x + 15$ **i** $f(x) = x^2 - 4x + 4$

2 Sketch graphs of the following quadratic functions.

a $y = x^2 - 9x + 20$ **b** $y = x^2 - 5x + 6$ **c** $y = x^2 - 13x + 12$
d $y = x^2 + 11x + 30$ **e** $y = x^2 + 5x + 4$ **f** $y = x^2 + 13x + 12$

g $y = x^2 - 4x - 12$ h $y = x^2 - x - 2$ i $y = x^2 - 5x - 14$
j $y = x^2 + 3x - 4$ k $y = x^2 + 7x - 30$ l $y = x^2 + 9x - 20$

3 Determine the x intercepts of the graphs of the following quadratic functions.

a $f(x) = 3x^2 + 10x + 3$ b $f(x) = 2x^2 + 3x + 1$
c $f(x) = 3x^2 - 5x + 2$ d $f(x) = 2x^2 - 11x + 5$
e $f(x) = 5x^2 + 2x - 3$ f $f(x) = 3x^2 + 11x - 4$
g $f(x) = 3x^2 - 2x - 1$ h $f(x) = 5x^2 - 24x - 5$

4 Determine the turning points of the following quadratic functions.

a $y = 2(x^2 - 7x + 10)$ b $y = 3(x^2 - 7x + 10)$
c $y = 4(x^2 - 6x + 8)$ d $y = -4(x^2 - 9x + 18)$
e $y = 3x^2 + 18x + 24$ f $y = 4x^2 + 24x + 32$
g $y = 3(x^2 - 81)$ h $y = -3(x^2 - 81)$
i $y = 3x^2 - 6x + 3$ j $y = 5x^2 - 10x + 5$

Example 7

5 Sketch graphs of the following quadratic functions.

a $f(x) = x^2 + 4x + 4$ b $f(x) = x^2 + 6x + 9$
c $f(x) = x^2 - 8x + 16$ d $f(x) = x^2 - 6x + 9$
e $f(x) = 2x^2 + 12x + 18$ f $f(x) = 3x^2 + 6x + 3$
g $f(x) = 3x^2 - 12x + 12$ h $f(x) = 5x^2 - 10x + 5$

6 Determine the coordinates of the turning point for these functions.

a $f(x) = x^2 - 2x + 1$ b $f(x) = x^2 + 8x + 16$
c $f(x) = x^2 - 10x + 25$ d $f(x) = 9x^2 - 6x + 1$
e $f(x) = 4x^2 + 20x + 25$ f $f(x) = 2x^2 + 4x + 2$

7 Sketch graphs of the following quadratic functions which include a difference of perfect squares.

a $y = x^2 - 4$ b $y = x^2 - 25$ c $y = x^2 - 16$
d $y = 3x^2 - 12$ e $y = 4x^2 - 4$ f $y = 25x^2 - 100$
g $y = 4x^2 - 9$ h $y = 9x^2 - 16$ i $y = 4x^2 - 121$

Enrichment

Th

8 For the following quadratic functions use a graph to help you determine the following using interval notation, e.g. $[-2, 3]$ or $(-\infty, -2) \cup (3, \infty)$:

i $\{x: f(x) = 0\}$ ii $\{x: f(x) < 0\}$ iii $\{x: f(x) > 0)$
iv $\{x: f(x) \leqslant 0\}$ v $\{x: f(x) \geqslant 0\}$

a $f(x) = x^2 + 7x + 12$ b $f(x) = x^2 - 8x + 16$
c $f(x) = 2x^2 + 3x - 20$ d $f(x) = 3x^2 + 11x + 8$
e $f(x) = x^2 + 8x + 16$ f $f(x) = x^2 - 12x + 36$
g $f(x) = 2x^2 - 50$ h $f(x) = 3x^2 - 75$

8.5 Sketching quadratic graphs using turning-point form

Quadratic functions can be written in the form $f(x) = a(x - h)^2 + k$ known as the **turning-point form**. This form is useful in that the turning point can be easily determined.

Key ideas

- By **completing the square,** all quadratic functions in the form $f(x) = ax^2 + bx + c$ can be expressed in turning-point form.
- In order to sketch a quadratic function in the form $f(x) = a(x - h)^2 + k$:
  - Determine the turning point (both its coordinates and nature).
    Turning point is at (h, k).
    If a is positive the parabola has a minimum turning point.
    If a is negative the parabola has a maximum turning point.
  - Determine the y intercept by substituting $x = 0$:

    $f(x) = a(0 - h)^2 + k$

  - Determine the x intercepts, if there are any, by substituting $y = 0$ and solving the equation using the null factor law if possible:

    $0 = a(x - h)^2 + k$

Example 8

For $f(x) = -4(x - 1)^2 + 16$:

a determine the coordinates and nature of its turning point
b determine the y intercept
c determine the coordinates of the x intercepts (if any)

Solution	Explanation
a Turning point is a maximum at (1, 16)	Turning point at (h, k) and $a < 0$
b y intercept at $x = 0$: $y = -4(0 - 1)^2 + 16$	Substitute $x = 0$
$= -4 + 16$	
$= 12$	
$\therefore$ y intercept at (0, 12)	
c x intercepts at $y = 0$:	Substitute $y = 0$
$0 = -4(x - 1)^2 + 16$	
$= (x - 1)^2 - 4$	Divide both sides by -4
$= (x - 1)^2 - (2)^2$	
$= (x - 1 - 2)(x - 1 + 2)$	Factorise using DOPS
$= (x - 3)(x + 1)$	
$\therefore x = 3$ or -1	Use the null factor law
x intercepts at $(-1, 0)$ and $(3, 0)$	

Example 9

Sketch the graph of $y = x^2 + 6x + 15$.

Solution

Turning point: $y = x^2 + 6x + 15$
$= x^2 + 6x + 9 - 9 + 15$
$= (x + 3)^2 + 6$

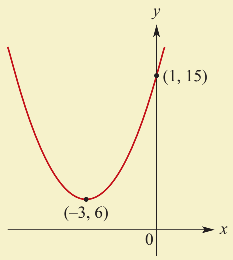
Turning point is minimum at $(-3, 6)$.

y intercept $x = 0$:
$y = (0)^2 + 6(0) + 15$
$= 15$

$\therefore y$ intercept at $(0, 15)$

x intercepts $y = 0$:
$0 = (x + 3)^2 + 6$

$\therefore$ no solution and no x intercepts



Explanation

Complete the square to change the equation into turning point form
Note: $a = 1$ in this case

Substitute $x = 0$

Substitute $y = 0$
This cannot be expressed as a difference of perfect squares (DOPS) and hence there are no factors and zero x intercepts

Sketch the graph showing the key points

Exercise 8E

Example 8a

1 Determine the coordinates and nature of the turning points of the following.

a $f(x) = 3(x - 2)^2 + 4$
b $f(x) = -4(x - 3)^2 + 6$
c $f(x) = -5(x - 3)^2 - 2$
d $f(x) = 9(x + 3)^2 - 4$
e $f(x) = 3(x + 8)^2 - 14$
f $f(x) = -4(x - 7)^2 + 2$
g $f(x) = -7(x - 2)^2 + 10$
h $f(x) = (x - 3)^2 + 6$

Example 8b

2 Determine the y intercepts of the following.

a $f(x) = (x + 2)^2 + 3$
b $f(x) = (x + 3)^2 - 7$
c $f(x) = (x - 1)^2 - 2$
d $f(x) = (x + 4)^2 - 5$
e $f(x) = -(x + 8)^2 - 14$
f $f(x) = -(x - 7)^2 + 2$
g $f(x) = x^2 + 6x + 3$
h $f(x) = x^2 + 5x + 1$
i $f(x) = x^2 + 7x - 5$
j $f(x) = x^2 + x - 5$
k $f(x) = x^2 - 5x + 12$
l $f(x) = x^2 - 12x - 5$

Example 8c

3 Determine the coordinates of x intercepts (if any) of the following.

a $f(x) = (x - 2)^2 - 9$
b $f(x) = (x + 4)^2 - 16$
c $f(x) = (x - 3)^2$
d $f(x) = (x + 7)^2$
e $f(x) = (x - 8)^2 + 25$
f $f(x) = (x - 7)^2 + 4$
g $f(x) = -(x - 2)^2 + 9$
h $f(x) = -(x - 3)^2 + 16$

4 Determine the x intercepts (if any) by first completing the square and rewriting the equation in turning-point form.

a $y = x^2 + 6x + 5$ **b** $y = x^2 + 6x + 1$ **c** $y = x^2 + 8x - 16$
d $y = x^2 + 2x - 8$ **e** $y = x^2 - 4x + 12$ **f** $y = x^2 - 12x - 13$

5 Sketch the graphs of the following.

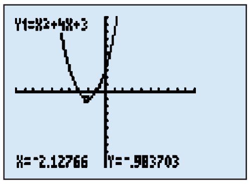
a $y = (x - 2)^2 - 4$ **b** $y = (x + 4)^2 - 9$ **c** $y = (x + 4)^2 - 1$
d $y = (x - 3)^2 - 4$ **e** $y = (x + 8)^2 + 16$ **f** $y = (x + 7)^2 + 2$
g $y = (x - 2)^2 + 1$ **h** $y = (x - 3)^2 + 6$ **i** $y = -(x - 5)^2 - 4$
j $y = -(x + 4)^2 - 9$ **k** $y = -(x + 9)^2 + 25$ **l** $y = -(x - 2)^2 + 4$

6 Sketch the graphs of the following quadratics functions.

a $f(x) = (x - 3)^2 - 2$ **b** $f(x) = (x + 5)^2 - 6$
c $f(x) = (x + 7)^2 + 3$ **d** $f(x) = (x - 6)^2 - 5$
e $f(x) = 4(x + 2)^2 + 12$ **f** $f(x) = 2(x + 7)^2 - 2$
g $f(x) = -3(x - 1)^2 + 15$ **h** $f(x) = -2(x + 3)^2 + 6$

7 Using technology, plot the graphs you have sketched in Question 6 and verify:

a the turning point
b the y intercept
c the x intercepts (if any)



Example 9

8 Sketch the graphs of the following quadratic functions.

a $y = x^2 + 4x + 3$ **b** $y = x^2 - 2x - 3$ **c** $y = x^2 + 6x + 9$
d $y = x^2 - 8x + 16$ **e** $y = x^2 - 2x - 8$ **f** $y = x^2 - 2x - 15$
g $y = x^2 + 8x + 7$ **h** $y = x^2 + 6x + 5$ **i** $y = x^2 + 12x$

9 Using technology, plot the graphs you have sketched in Question 8 and verify:

a the turning point **b** the y intercept **c** the x intercepts (if any)

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Enrichment

10 Sketch the parabola that represents the following quadratic functions. (First take out the coefficient of x^2 and introduce brackets.)

a $y = 4x^2 + 8x + 3$ **b** $y = 3x^2 - 12x + 10$ **c** $y = 2x^2 + 12x - 1$
d $y = 2x^2 + x - 3$ **e** $y = 2x^2 - 7x + 3$ **f** $y = 4x^2 - 8x + 20$
g $y = 6x^2 + 5x + 9$ **h** $y = 5x^2 - 3x + 7$ **i** $y = 5x^2 + 12x$
j $y = 7x^2 - 10x$

11 Using technology, plot the graphs you have sketched in Question 10 and verify:

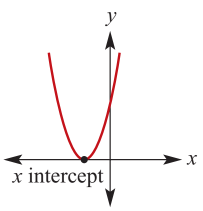
a the turning point **b** the y intercept **c** the x intercepts (if any)

8.6 Sketching quadratic graphs using the quadratic formula

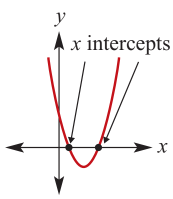
If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant $\Delta = b^2 - 4ac$ determines the number of solutions to the equation.

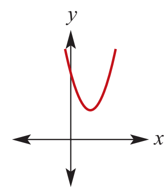
If $\Delta = \mathbf{0}$ then $b^2 - 4ac = 0$. The solution to the equation becomes $x = -\dfrac{b}{2a}$ i.e. 1 solution and 1 x intercept



If $\Delta > \mathbf{0}$ then $b^2 - 4ac > 0$. The solution to the equation becomes $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ i.e. 2 solutions and 2 x intercepts



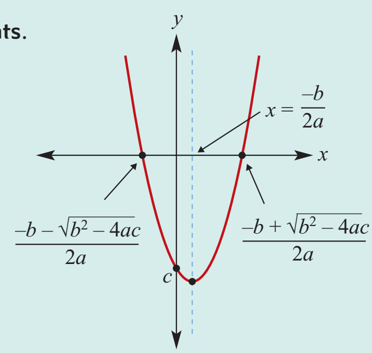
If $\Delta < \mathbf{0}$ then $b^2 - 4ac < 0$. Square roots exist for positive numbers only, i.e. 0 solutions or x intercepts



Key ideas

To sketch the graph of $y = ax^2 + bx + c$, find the following points.

- **y intercept** at $x = 0$: $y = a(0)^2 + b(0) + c = c$
  $\therefore$ y intercept at $(0, c)$
- **x intercepts** when $y = 0$: $0 = ax^2 + bx + c$
  $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}, \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$
  (0, 1 or 2 solutions and 0, 1, or 2 intercepts)
  If $\Delta < 0 \rightarrow 0$ x intercepts
  If $\Delta = 0 \rightarrow 1$ x intercept
  If $\Delta > 0 \rightarrow 2$ x intercepts



- **turning point:** The x coordinate lies halfway between the x intercepts so $x = -\dfrac{b}{2a}$
  The y coordinate is found by substituting the x coordinate into the original function.

Example 10

For the parabola given by the quadratic function $f(x) = 3x^2 - 6x + 5$:

a determine the number of x intercepts

b determine the y intercept

c using $x = \dfrac{-b}{2a}$ determine the turning point

Solution

a $b^2 - 4ac = (-6)^2 - 4(3)(5)$
$\quad\quad = -14$
$\Delta < 0 \therefore 0$ x intercepts

b y intercept at $(0, 5)$

c $x = -\dfrac{b}{2a} = -\dfrac{(-6)}{2(3)} = 1$

$y = 3(1)^2 - 6(1) + 5$
$\quad = 3 - 6 + 5 = 2$
$\therefore$ turning point at $(1, 2)$

Explanation

Use the discriminant to determine the number of x intercepts

y intercept is at $(0, c)$

Determine the x value for the turning point

Determine the y value of the turning point

Write the coordinates of the turning point

Example 11

Sketch the graph of the quadratic function $y = 2x^2 + 4x - 3$, labelling all significant points. Round the x intercepts to two decimal places.

Solution

***y* intercept** at $(0, -3)$

***x* intercepts**: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-4 \pm \sqrt{4^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{-4 \pm \sqrt{40}}{4}$$

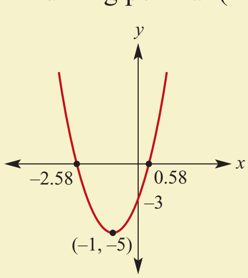
$$= \frac{\cancel{4}^{2} \pm \cancel{2}^{1}\sqrt{10}}{\cancel{4}_{2}} = \frac{2 \pm \sqrt{2}}{2}$$

$x \approx 0.58, -2.58$

$\therefore$ x intercepts at $(0.58, 0)$ and $(-2.58, 0)$

Turning point is at $x = -\dfrac{b}{2a} = -\dfrac{(4)}{2(2)}$
$\quad = -1$

and $\therefore$ $y = 2(-1)^2 + 4(-1) - 3$
$\therefore$ turning point at $(-1, -5)$



Explanation

y intercept is at $(0, c)$

Use the quadratic formula to solve for the x intercepts

Write the coordinates of the intercepts

Determine the x value for the turning point

Determine the y value of the turning point

Write the coordinates of the turning point

Sketch the graph

Exercise 8F

Example 10a

1 Determine the number of x intercepts for the parabolas given by the following quadratic functions.

a $f(x) = x^2 + 2x + 1$ **b** $f(x) = x^2 - 3x + 4$ **c** $f(x) = -x^2 + 4x - 2$
d $f(x) = -3x^2 + 4x + 6$ **e** $f(x) = 2x^2 - x + 2$ **f** $f(x) = 2x^2 - 12x + 18$
g $f(x) = 3x^2 - 2x$ **h** $f(x) = 4x^2 + 6x$ **i** $f(x) = -3x^2 + 2x$
j $f(x) = 4x^2 + 3$ **k** $f(x) = 3x^2 - 3$ **l** $f(x) = -2x^2 + x$

Example 10b

2 Determine the y intercept for the parabolas given by the following quadratic functions.

a $f(x) = x^2 + 2x + 5$ **b** $f(x) = x^2 - 3x + 2$ **c** $f(x) = -x^2 + 3x - 2$
d $f(x) = 3x^2 + 2x - 1$ **e** $f(x) = -6x^2 - 6x + 5$ **f** $f(x) = -2x^2 + 9x - 7$
g $f(x) = 2x^2 + 5x - 15$ **h** $f(x) = -4x^2 - 3x$ **i** $f(x) = -3x^2 - 9$

Example 10c

3 Using $x = \dfrac{-b}{2a}$, determine the turning points for the parabolas defined by the following quadratic functions.

a $f(x) = x^2 - 2x + 4$ **b** $f(x) = -x^2 + 2x - 1$
c $f(x) = x^2 + 3x + 1$ **d** $f(x) = -x^2 + 3x - 4$
e $f(x) = -x^2 - 3x + 4$ **f** $f(x) = -x^2 + 7x - 7$
g $f(x) = 2x^2 + 3x - 4$ **h** $f(x) = 4x^2 - 3x$
i $f(x) = -3x^2 - 9$ **j** $f(x) = -4x^2 + 2x - 3$
k $f(x) = -3x^2 - 2x$ **l** $f(x) = -7x^2 - 4$

Example 11

4 Sketch completely the parabolas given by the following quadratic functions. Round any x intercepts to two decimal places.

a $y = 2x^2 + 6x - 5$ **b** $y = 3x^2 - 6x - 1$ **c** $y = 2x^2 + 4x + 3$
d $y = 2x^2 + 8x + 3$ **e** $y = 3x^2 + 3x - 10$ **f** $y = 2x^2 + 6x + 5$
g $y = 3x^2 - 5x - 1$ **h** $y = 2x^2 + 7x - 2$ **i** $y = 2x^2 + 5x - 12$

5 Sketch the parabolas given by the following quadratic functions, labelling all significant points.

a $y = x^2 + 5x + 6$ **b** $y = x^2 - 4x + 3$ **c** $y = x^2 - 3x + 3$
d $y = x^2 + 2x + 2$ **e** $y = x^2 + 4x + 4$ **f** $y = x^2 - 6x + 9$
g $y = -x^2 + 3x - 4$ **h** $y = -x^2 - 5x + 6$ **i** $y = -x^2 - 8x - 16$
j $y = -x^2 + 2x - 1$ **k** $y = -x^2 + 3x - 1$ **l** $y = -x^2 - 2x + 4$

Enrichment

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6 Substitute $x = -\frac{b}{2a}$ into $y = ax^2 + bx + c$ to find the general rule for the y coordinate of the turning point in terms of a, b and c.

7 Prove the rule $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by solving $ax^2 + bx + c = 0$. *Hint:* Divide both sides by a and complete the square.

8.7 Applications of quadratics

Many real-life objects or situations can be modelled by quadratic functions including lenses, satellite dishes, parts of suspension bridges and projectiles.

It is often the case that key features of a quadratic sketch can help to determine the solution to problems that are modelled by quadratic functions.

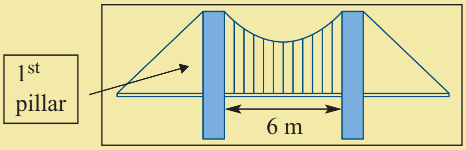
Key ideas

When using quadratic models and their graphs we should consider:

- the domain of the function (which can be stated, for example, as $-5 \leqslant x < 7$ or $[-5, 7)$)
- the coordinates of the endpoints of the graph
- the turning point of the graph
- any axis intercepts

Example 12

A civil engineer designs a model of a 6 m suspension bridge using the equation $h = (d - 3)^2 + 7$, where h is the height of the hanging cables above the road for a distance d from the first pillar.



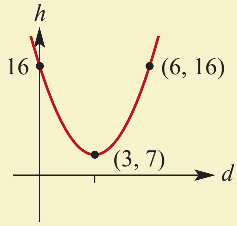
a What are the possible values for d?

b Sketch the graph of $h = (d - 3)^2 + 7$ over the appropriate domain.

c What is the height of the pillars above the road?

d If the vertical cables are hung every 0.5 m along the bridge, use a graphics or CAS calculator to determine the length of the cable needed:

i 2 m from the pillar **ii** 3.5 m from the pillar

Solution	Explanation
a $0 \leqslant d \leqslant 6$	The pillars are 6 m apart
b 	Sketch the graph showing the coordinates of the endpoints
c $h = (0 - 3)^2 + 7 = 16$	Substitute $d = 0$ or $d = 6$
d **i** $h = (2 - 3)^2 + 7 = 8$ $\therefore$ length $=$ 8 metres	Substitute $d = 2$ using VALUE or the TABLE
ii $h = (3.5 - 3)^2 + 7 = 7.25$ $\therefore$ length $=$ 7.25 metres	Substitute $d = 3.5$

Exercise 8G

Example 12

1 A ball is thrown upwards from ground level and reaches a height of h metres after t seconds, given by the formula $h = 16t - 4t^2$.

a What are the possible values for t?

b Sketch the graph of h versus t over the appropriate domain.

c What maximum height does the ball reach?

d How long does it take the ball to return to ground level?

e Using a graphics or CAS calculator, determine at what times the ball is at a height of:

i 20 m **ii** 10 m

2 The path of a javelin thrown by Marita is given by the formula $h = -\frac{1}{16}(d - 10)^2 + 9$ where h m is the height of the javelin above the ground and d m is the horizontal distance travelled.

a What are the possible values of d?

b Sketch the graph over the appropriate domain (possible d values).

c What is the maximum height the javelin reaches?

d What distance does the javelin travel?

e Using a graphics or CAS calculator, experiment to find which numbers in the equation would need to change and how they would differ if:

i the javelin was thrown higher

ii the javelin thrower was taller than Marita

iii the distance travelled was less overall

3 A wood turner hones out a bowl according to the formula $d = \frac{1}{3}x^2 - 27$ where d cm is the depth of the bowl and x cm the distance from the centre of the bowl.

a Sketch the graph over an appropriate domain.

b What is the width of the bowl?

c What is the maximum depth of the bowl?

d Using a graphics or CAS calculator, determine:

i the approximate distance away from the centre when $d = -2.5$

ii d, when the distance from the centre is 7 cm

4 The equation for the arch of a particular bridge is given by the equation $h = -\frac{1}{500}(x - 100)^2 + 20$ where h m is the height above the base of the bridge and x m is the distance from the left side.

a Determine the turning point of the graph of the relationship.

b Determine the x intercepts of the graph of the relationship.

c Determine the possible values that x can take.

d Sketch the graph of the arch over an appropriate domain.

e What is the span of the arch?

f What is the maximum height of the arch?

g If a lookout is to be located on the arch at a height of 10 m, use a graphics or CAS calculator to determine where it would be located on the arch.

5 The equation for a support span is given by $h = -\frac{1}{40}(x - 20)^2$ where h m is the distance below the base of a bridge and x m is the distance from the left side.

- **a** Determine the turning point of the graph of the relationship.
- **b** Determine the possible values of x.
- **c** Determine the range of values of h.
- **d** Sketch a graph of the equation over the appropriate domain.
- **e** What is the width of the support span?
- **f** What is the maximum height of the support span?

6 The height of a parachutist above the ground (h m) is represented by the relationship

$$h = -40t^2 + 4t + 10\,000$$

where t is the time in minutes after jumping from the aeroplane.

- **a** Sketch a graph of h versus t over an appropriate domain.
- **b** How long does it take the parachutist to reach the ground?
- **c** At what height did the parachutist jump from the plane?

7 A clay pigeon is projected from a platform and a shooter tries to hit it before it hits the ground to score points in a competition. The height of the clay pigeon above the ground (h m) is given by the equation

$$h = -2t^2 + 9t + 3$$

where t is the time, in seconds, after the clay pigeon is released.

- **a** Sketch a graph of h versus t over an appropriate domain.
- **b** What is the maximum height reached and when did it happen?
- **c** What is the height of the platform?
- **d** How much time does the shooter have to hit the clay pigeon?

Th

Enrichment

8 A car travels on a stretch of farm road modelled by the equation $y = x^2 - 4x - 12$ where $-3 \leqslant x \leqslant 8$.

- **a** Sketch a graph over the given domain.
- **b** Determine the equation of a straight line that joins the endpoints of this model.
- **c** A fence is to be built on a line modelled by the equation $y = 2x - 5$. At what coordinates will the farmer need to install gates?

8.8 Modelling with quadratics

Creating models involves setting up a function that satisfies the given conditions. All key ideas from the last section apply except you will need to set up your own equations and pay particular attention that the domain suits the situation you are modelling.

Key ideas

Modelling with quadratics may involve:

- defining variables
- forming equations
- eliminating a variable
- defining a suitable domain
- sketching a graph over an appropriate domain
- finding maximums and minimums or other key elements of a function and its graph

Example 13

A piece of wire 100 cm in length is bent into the shape of a rectangle.

a Determine y in terms of x.

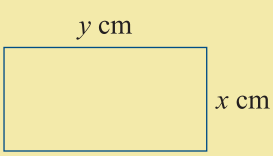
b Write an equation for the area of the rectangle (A) in terms of x.

c Find the domain of $A(x)$.

d Sketch the graph of A versus x for suitable values of x.

e Use the graph to determine the maximum area that can be formed.

f What will the dimensions of the rectangle be to achieve its maximum area?



Solution

a Perimeter $= 2x + 2y = 100$

$$2y = 100 - 2x$$

$$y = 50 - x$$

Explanation: 100 m of wire is used, so the perimeter of the rectangle is 100 cm

b $A = xy$

$= x(50 - x)$

Explanation: Area of a rectangle is lenght × width. Replace y with $50 - x$

c $x > 0$ and $y > 0$

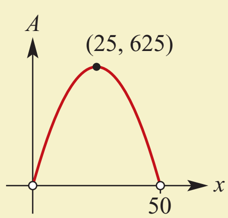
So $-x > 0$

$x < 50$

$\therefore$ Domain $= (0, 50)$

Explanation: The domain of the function is (0, 50) as both lengths must be positive

d



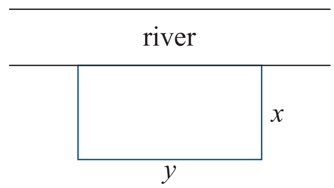
Explanation: First find the x intercepts then show the coordinates of the turning point

e	The maximum area that can be formed is 625 cm^2.		Reading from the turning point of the graph
f	$x = 25$ $y = 50 - 25 = 25$ The dimensions of the rectangle are 25 cm and 25 cm.		The dimensions are defined by x and y and $y = 50 - x$

Exercise 8H

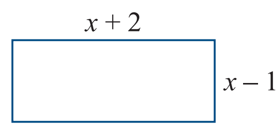
Example 13

1 A farmer has 100 m of fencing to form a paddock as shown.



- **a** Determine y in terms of x.
- **b** Write an equation for the area of the paddock (A) in terms of x.
- **c** Find the domain of $A(x)$.
- **d** Sketch the graph of A versus x for suitable values of x.
- **e** Use the graph to determine the maximum area of the paddock that can be formed.
- **f** What will the dimensions of the paddock be to achieve its maximum area?

2 The dimensions of a rectangular garden are represented by $(x + 2)$ m and $(x - 1)$ m.

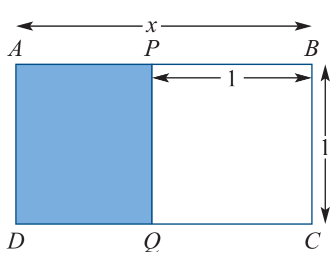


- **a** Write an equation for the area of the garden $A(x)$ in terms of x.
- **b** Find the value of x if the area of the garden is 28 m^2.
- **c** **i** Find the domain of $A(x)$.
  - **ii** Sketch the graph of A versus x.
  - **iii** Determine the maximum area if $1 < x < 20$.

3 The dimensions of a rectangular children's playground are such that its length is 3 m longer than its width.

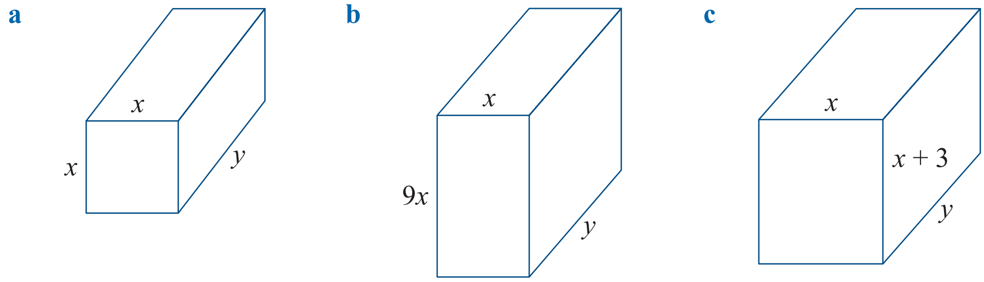
- **a** Write an equation for the area $A(x)$ of the playground.
- **b** If the area of the playground is 70 m^2, determine its length.
- **c** **i** Find the domain of $A(x)$.
  - **ii** Sketch the graph of A versus x.
  - **iii** Determine the maximum area if $0 < x < 20$.

4 The golden rectangle has fascinated architects and artists over the centuries. It is perceived to be the rectangle that is the most pleasant to the eye.
If a square is drawn on one of the longer sides, then the new rectangle is similar to the original. If $AB = x$:



- **a** determine the length of AP in terms of x
- **b** write the similar ratios $\dfrac{AB}{BC} = \dfrac{AD}{AP}$ in terms of x
- **c** rearrange the equation from part **b** to equal zero
- **d** determine the value of x, known as the golden ratio

5 Three cuboids are constructed from some plastic tubing of length 64 cm.



i For each of the cuboids, write equations for the surface area ($S(x)$) of the top surface in terms of x and y.

ii For each of the cuboids, write equations for the length of tubing needed to construct it ($L(x)$) in terms of x and y.

iii If the length of the tubing needed to construct the shapes is 64 cm, determine y in terms of x.

iv Determine the possible values for x.

v Rewrite $S(x)$ in terms of x.

vi Sketch the graph of $S(x)$ versus x (a graphics or CAS calculator may be used).

vii Determine the maximum surface area of the top and the dimensions of the box.

Enrichment

Th

6 A section of roller coaster takes the shape of a parabola and can be modelled by a quadratic function for $x \in [85, 105]$. The coordinates of the start and finish of the section are (85, 0) and (105, 0) respectively.

a Given that the model is of the form $f(x) = A(x - a)(x - b)$, determine an equation for the following if the maximum height reached is:

i 100 m

ii 75 m

iii 125 m

Vertical beams at specific points support the track.

b Determine the height of a beam needed to support the track when $x = 95$ m.

c Two vertical support beams have been cut at 50 metres in length. Determine the positions on the track they need to be placed to the nearest whole number. (A graphics or CAS calculator may be used.)

WORKING Mathematically

Quadratic graphs

PL
E
KT
Th

1 Flight paths and archery

An archer has a powerful bow that he can fire accurately over 200 metres. He estimates that he would need to angle his bow so that it would reach a maximum height of 80 metres if he wished to hit the target 200 metres away.

As the path of the arrow is parabolic in shape, it can be modelled by a quadratic equation.

In order to find this equation, we will use turning-point form.

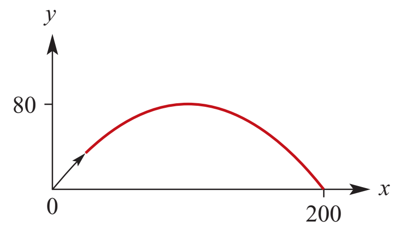
$y = a(x - h)^2 + k$

Turning point at (100, 80)

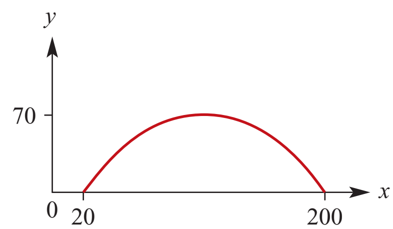
$\therefore y = a(x - 100)^2 + 80$

Now (200, 0) is on the curve

$$\therefore 0 = a(200 - 100)^2 + 80$$
$$-80 = a(100)^2$$
$$\frac{-80}{10\,000} = a$$
$$-0.008 = a$$
$$\therefore y = -0.008(x - 100)^2 + 80$$



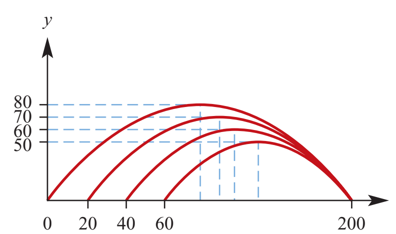
Determining an equation for the flight path



The archer estimates that, as he moves towards the target, the height his arrow has to reach decreases. In fact, when he moves 20 metres closer the maximum height is 70 metres.

Determine the equation of the path of the arrow after he has moved 20 metres closer to the target.

Closer and closer



The archer estimates that this pattern will continue. Every 20 metres he moves towards the target will decrease by 10 metres the height the arrow needs to reach for the target to be hit.

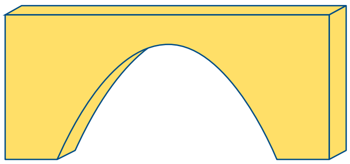
a Sketch on the same axes the pathways of the arrows as the archer moves towards the target in 20-metre intervals.

b At what point do you estimate that the path no longer needs to take a parabolic shape?

c At this position, suggest another appropriate model for the path of the arrow.

d Determine the equations of the path of the arrows is fired from 20, 40, 60 . . . metres closer than the original position.

2 Painting bridges



A bridge is 6 m high and has an overall width of 12 m and an archway underneath as shown.

We need to determine the actual surface area of the face of the arch so that we can order paint to refurbish it.

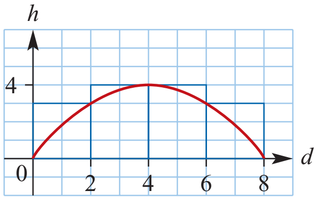
The equation for this area would be:

bridge area $= 12 \times 6 -$ area under arch
$= 72 -$ area under the parabola

Consider an archway modelled by the formula $h = -\frac{1}{4}(d - 4)^2 + 4$.

To estimate the area under the arch, divide the area into rectangular regions.

If we draw rectangles above the arch and calculate there areas we will have an estimate of the area under the arch even though it is slightly too large.



Area $= (2 \times 3) + (2 \times 4) + (2 \times 4) + (2 \times 3)$
$= 6 + 8 + 8 + 6 = 28\ \text{m}^2$

$\therefore$ Area is approximately $28\ \text{m}^2$

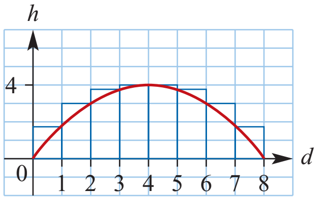
We could obtain a more accurate answer by increasing the number of rectangles, i.e. reducing the width of each rectangle (called the strip width).

Overestimating the area under the arch

a Construct an accurate graph of the parabola and calculate the area underneath using a strip width of 1.

b Repeat your calculations using a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.

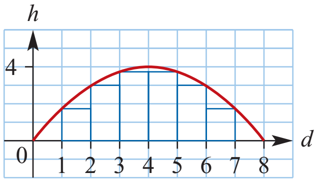


Underestimating the area under the arch

a Estimate the area under the arch by drawing rectangles beneath the graph with a strip width of 1.

b Repeat the process for a strip width of 0.5.

c Calculate the surface area of the face of the arch using your answer from part **b**.



Improving accuracy

a Suggest how the results from parts 1 and 2 could be combined to achieve a more accurate result.

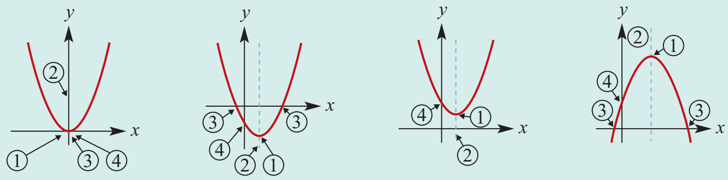
b Explore how a graphics or CAS calculator can give accurate results for finding areas under curves.

Essential Mathematics VELS Projects

Chapter summary

Quadratic equations and parabolas

$f(x) = ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$



The key features of a parabola are:

1 turning point: a maximum or a minimum
2 axis of symmetry
3 x intercepts (can be 0, 1 or 2 intercepts)
4 y intercept

Transformations

- dilation—making the graph wider or narrower
- translation—shifting the graph left or right and up or down
- reflection—across the x and y axes

Sketching a parabola from transformations

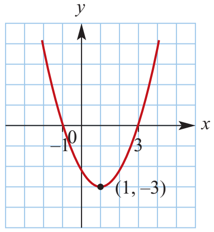
Perform transformations in the correct order, dilation and reflection then translation. We should show the following key features:

- the shape—to show maximum or minimum
- the turning point—to show the translation of $y = x^2$
- any other point (usually the y intercept or when $x = 1$ if the y intercept is at the turning point)—to show the dilation of $y = x^2$

Summary	$y = ax^2 + bx + c$ where we can factorise by inspection or table	$y = a(x - h)^2 + k$	$y = ax^2 + bx + c$ using the quadratic formula
y intercept	at $(0, c)$	substitute $x = 0$: $y = a(0 - h)^2 + k$	at $(0, c)$
x intercepts	Factorise the equation and use the null factor law to determine the x intercepts. If there is only one x intercept it will occur at the turning point of the graph.	at $y = 0$: $0 = a(x - h)^2 + k$ and solving the equation using the null factor law	at $y = 0$: $0 = ax^2 + bx + c$ $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ (0, 1 or 2 solutions or 0, 1, or 2 intercepts) If $\Delta < 0$ 0 x intercepts If $\Delta = 0$ 1 x intercept If $\Delta > 0$ 2 x intercepts
Turning point	The x ordinate: half way between the x intercepts if there are 2 At the x intercept if there is 1 Substitute the x ordinate into the equation $y = ax^2 + bx + c$ to determine the y ordinate	at (h, k) If a is positive: minimum If a is negative: maximum	$x = -\frac{b}{2a}$ y is found by substituting x into the formula $y = ax^2 + bx + c$. If a is positive: minimum If a is negative: maximum

Multiple-choice questions

Questions 1 to 5 refer to the following graph



1 The turning point of the graph is

A $x = 1$ **B** $x = -1$ **C** $x = 3$ **D** $(1, -3)$ **E** $(-1, 3)$

2 The coordinates of the y intercept are

A $y = -2$ **B** $(-2, 0)$ **C** $(0, -2)$ **D** $(1, -3)$ **E** $x = 0$

3 The axis of symmetry is

A $y = -2$ **B** $x = -1$ **C** $x = 3$ **D** $x = 1$ **E** $x = -3$

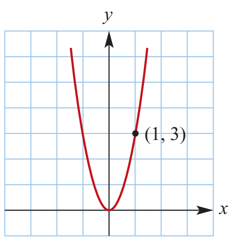
4 The coordinates of the x intercepts are

A $(-1, 0)(3, 0)$ **B** $(1, -3)(-2, 0)$ **C** $(0, -1)(0, 3)$
D $(-3, 1)(0, -2)$ **E** $(-1, -2)(1, -3)$

5 Which of the following statements is not true about the graph?

A It is a parabola. **B** It has 2 x intercepts. **C** It is a function.
D It passes through the origin. **E** It has a minimum turning point.

Questions 6 and 7 refer to the following graph



6 The equation of the graph is

A $f(x) = (x - 1)^2 + 3$ **B** $f(x) = 3x^2$ **C** $f(x) = (x + 1)^2 + 3$
D $f(x) = 3x^2 + 1$ **E** $f(x) = (x - 3)^2 + 1$

7 The transformation of $y = x^2$ to the graph shown can be described as

A a translation of 3 across and 2 up **B** a translation of 1 across and 3 up
C a dilation of 3 upwards **D** a translation of 1 in the x direction
E a dilation of 3 in the y direction

8 If the graph of $y = x^2$ is translated 3 units downward it

A has a maximum turning point **B** has 0 x intercepts
C has 1 x intercept **D** has 2 x intercepts
E has 2 y intercepts

9 The graph of $f(x) = 3x^2 + 2x - 2$ has

A 0 x intercepts **B** 1 x intercept **C** 2 x intercepts

D 0 y intercepts **E** 2 y intercepts

10 The turning point for the parabola given by the quadratic equation $y = -x^2 - 3x + 4$ is

A $(-\frac{3}{2}, 6\frac{1}{4})$ **B** $(-\frac{3}{2}, -2\frac{3}{4})$ **C** $(-\frac{3}{2}, 10\frac{1}{4})$ **D** $(\frac{3}{2}, 6\frac{1}{4})$ **E** $(\frac{3}{2}, 10\frac{1}{4})$

Short-answer questions

1 Plot the graphs of the following, showing the key features.

a $y = x^2 + 5$ **b** $y = (x - 3)^2 + 1$ **c** $y = x^2 + 2x - 3$

2 For $f(x) = x^2 + 5x - 4$ determine:

a $f(0)$ **b** $f(3)$ **c** $f(-4)$ **d** $f(-6)$

3 Find the coordinates of the endpoints of $f(x) = 3(x - 2)^2 + 4$ for the given domains.

a $1 \leqslant x \leqslant 5$ **b** $-3 < x < 1$

4 Find the maximum value of $f(x)$ if:

a $f(x) = (x - 2)^2 + 1 \quad 0 \leqslant x \leqslant 1$ **b** $f(x) = -3x^2 + 12 \quad 2 \leqslant x \leqslant 4$

5 Sketch the graphs of the following.

a $y = 4x^2$ **b** $y = -5x^2$ **c** $y = x^2 + 4$ **d** $y = -(x - 3)^2$

6 Write equations that result from the transformation of $y = x^2$ by being:

a translated 3 units left and 2 units up

b reflected in the x axis and translated 4 units down

c reflected in the x axis and translated 4 units down

7 Sketch the graphs of the following functions, showing the turning point and any intercepts.

a $f(x) = x^2 - 7x + 10$ **b** $f(x) = 2x^2 + 6x + 4$ **c** $f(x) = x^2 - 25$

d $f(x) = 14x^2 - 56$ **e** $f(x) = x^2 + 14x + 49$ **f** $f(x) = 3x^2 - 36x + 108$

8 Sketch the graphs of the following functions, showing the turning point and any intercepts.

a $f(x) = (x - 3)^2 - 4$ **b** $f(x) = -(x - 2)^2 + 9$

9 By first completing the square and rewriting the equation in turning-point form, determine:

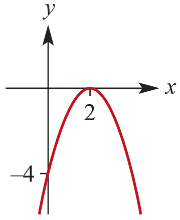
i the turning point **ii** x intercepts (if any)

a $y = x^2 + 8x - 16$ **b** $y = x^2 - 12x - 13$

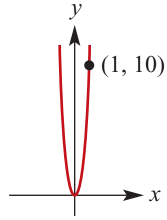
10 **i** Determine the equations of the following graphs.

ii Describe the transformations of $y = x^2$ required to change to these graphs.

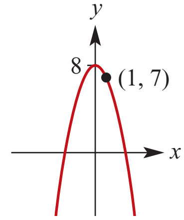
a



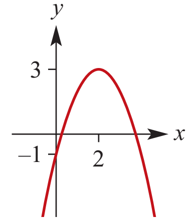
b



c



d



11 Use the quadratic formula to sketch the parabolas given by the following quadratic equations, labeling all significant points.

a $y = x^2 + 2x - 2$

b $y = -x^2 + 3x + 4$

12 The dimensions of a rectangular are such that its length is 15 m longer than its width (x).

a Write an equation for the area of the rectangle $A(x)$.

b If the area of the rectangle is 100 m^2, determine its length.

13 a Plot the graph of the function $f(x) = x^2 + 5x - 6$.

b List the coordinates of:

i the x intercepts ii the y intercept iii the turning point

c Determine:

i $f(0)$ ii $f(6)$

d Determine x when:

i $f(x) = 0$ ii $f(x) = 8$

Extended-response questions

1 The cable for a suspension bridge is modelled by the equation

$$h = \frac{1}{1500}(x - 300)^2 + 20$$

where h m is the distance above the base of the bridge and x m is the distance from the left side of the bridge.

a Determine the turning point of the graph of the relationship.

b Determine the possible values of x.

c Determine the range of values of h.

d Sketch a graph of the equation over the appropriate domain.

e What distance does the cable span?

f What is the closest distance the cable is from the base of the bridge?

g What is the greatest distance the cable is from the base of the bridge?

2 An open rectangular box is 2 cm wider than it is high and twice as long as it is wide. If x cm is the height of the box:

a find an expression for its total surface area $A(x)$ cm^2 in terms of x

b find $A(x)$ if:

i $x = 1$ ii $x = 2$

c find the value of x if $A(x) = 386$

d sketch the graph of A versus x for suitable values of x

e using a graphics calculator and its table of values determine the value of x (to 1 decimal place) if the block has a surface area of?

i 150 cm^3 ii 1000 cm^3

Review

MC TEST

D&D TEST